Representations of the braid group B_n and the highest weight modules of $U(\mathfrak{sl}_{n-1})$ and $U_q(\mathfrak{sl}_{n-1})$

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Abstract

In [1] we have constructed a $\left[\frac{n+1}{2}\right]+1$ parameters family of irreducible representations of the Braid group B_3 in arbitrary dimension $n\in\mathbb{N}$, using a $q-$deformation of the Pascal triangle. This construction extends in particular results by S.P. Humphries (2000), who constructed representations of the braid group B_3 in arbitrary dimension using the classical Pascal triangle. E. Ferrand (2000) obtained an equivalent representation of B_3 by considering two special operators in the space $\mathbb{C}^n[X]$. Slightly more general representations were given by I. Tuba and H. Wenzl (2001). They involve $[\frac{n+1}{2}]$ parameters (and also use the classical Pascal's triangle). The latter authors also gave the complete classification of all simple representations of B_3 for dimension $n\leq 5$. Our construction generalize all mentioned results and throws a new light on some of them. We also study the irreducibility and equivalence of the constructed representations.

In the present article we show that all representations constructed in [1] may be obtained by taking exponent of the highest weight modules of $U(\mathfrak{sl}_2)$ and $U_q(\mathfrak{sl}_2)$. *We generalize these connections* between the representation of the braid group B_n and the highest weight modules of the $U_q(\mathfrak{sl}_{n-1})$ *for arbitrary* n using the well-known *reduced Burau representations.*

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1 Introduction. Braid group representations

Our *aim* is to describe the *dual* $\hat{B}_n$ of the *braid group* B_n. It is natural to compare the *representation theory* of the *symmetric group* S_n and of the braid group B_n. We know almost everything about representation theory of the symmetric group S_n. We know the description of the *dual* $\hat{S}_n$ in terms of *Young diagrams*. We know even the *Plancherel measure* on $\hat{S}_n$. The *Young graph* explains how to decompose the restriction $\pi\mid_{S_{n-1}}$ of the representation $\pi \in \hat{S}_n$, etc.

The braid groups B_n are *defined* by the generators σ_i, $1 \leq i \leq n-1$ and by the relations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, $\quad \sigma_i\sigma_j = \sigma_i\sigma_j$ for $\mid i-j \mid \geq 2$. The *dual* $\hat{B}_n$ of the group B_n *is known* only for the *commutative case* when $n = 2$. In this case $B_2 \cong \mathbb{Z}$ hence $\hat{B}_2 \cong S^1$. The *representation theory* for the braid groups B_n is much more *complicated* than for S_n. The *reason* is the following. In the case of the group S_n we have the essential (*quadratic) relation* $\sigma_i^2 = 1$, hence $Sp\,(\pi(\sigma_i)) \subseteq \{-1, 1\}$. In the case of the group B_n we do not have these conditions. Since $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ we have $Sp\,(\pi(\sigma_i)) = Sp\,(\pi(\sigma_{i+1}))$, but the *spectra* $Sp\,(\pi(\sigma_i))$ may be almost *arbitrary*.

The *Hecke algebra* $H_n(q)$ see f.e.[15] appears as the factor algebra of the group algebra of the group B_n subject to the following *quadratic relation* $\sigma_i^2 = (q-1)\sigma_i + q$, $1 \leq i \leq n-1$, hence $Sp\,(\pi(\sigma_i)) \subseteq \{-1, q\}$ and $H_n(q) \cong \mathbb{C}[S_n]$. This is a reason why the representation theory of Hecke algebras is well developed.

The *next step* is to impose the *polynomial condition* $p_k(\sigma_i) = 0$ on the generators σ_i where k is the order of the polynomial $p_k(x)$. For $k = 3$ the corresponding algebra is called *Birman–Murakami–Wenzl type algebra* or simple BMW algebra see [26, 32] (see also [27]) and so on.

The situation becomes much more complicated if no additional conditions on the spectra are imposed. We *shall study* this *general case* for .

In [29] I.Tuba and H.Wenzl gave the *complete classification* of all *simple representations* of B_3 for *dimension* ≤ 5.

In [12] E.Formanek et al. gave the *complete classification* of all *simple representations of* B_n for *dimension* $\leq n$.

We *generalize the results* I.Tuba and H.Wenzl for B_3, give *new representations* of B_n for *large dimension* and establish *connection* between the *representations* of B_n and *the highest weight modules* of the *quantum group*

$U_q(\mathfrak{sl}_{n-1})$.

More precisely, in the work [1] with S.Albeverio we have constructed a $\left[\frac{n+1}{2}\right]+1$ parameter family of irreducible representations of the braid group B_3 it in arbitrary dimension $n\in\mathbb{N}$, using *a $q-$deformation of the Pascal triangle*. This construction extends in particular results by S.P. Humphries [14], I. Tuba and H. Wenzl [29], and E. Ferrand [11]. The *irreducibility* and the *equivalence* of the constructed representations is studied. For example the representations corresponding to different q and n are *nonequivalent*.

In this article we show that there is a striking *connection* between these *representations* of B_3 and a highest weight modules of the *quantum group* $U_q(\mathfrak{sl}_2)$, a one-parameter *deformation of the universal enveloping algebra* $U(\mathfrak{sl}_2)$ of the Lie algebra $\mathfrak{sl}_2$. The starting point for all these considerations is some homomorphism ρ_3 of the braid group B_3 into $\mathrm{SL}(2,\mathbb{Z})$:

$$\rho_3: B_3\mapsto \mathfrak{sl}_2 \overset{\exp}{\mapsto} \mathrm{SL}(2,\mathbb{Z})$$

$$\sigma_1\mapsto \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right) \overset{\exp}{\mapsto} \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right),\ \sigma_2\mapsto \left(\begin{smallmatrix}0&0\\-1&0\end{smallmatrix}\right) \overset{\exp}{\mapsto} \left(\begin{smallmatrix}1&0\\-1&1\end{smallmatrix}\right).$$

The constructed representations may be treated as the *$q-$symmetric power* of this *fundamental representation* or as an appropriate *$q-$exponential* of the highest weight modules of $U_q(\mathfrak{sl}_2)$.

We generalize these connections between the representation of the braid group B_n and the highest weight modules of the $U_q(\mathfrak{sl}_{n-1})$ *for arbitrary n* using the well-known *reduced Burau representation* $b_n^{(t)}$ see c.f. [15]. We note that in particular $\rho_3=b_3^{(-1)}$.

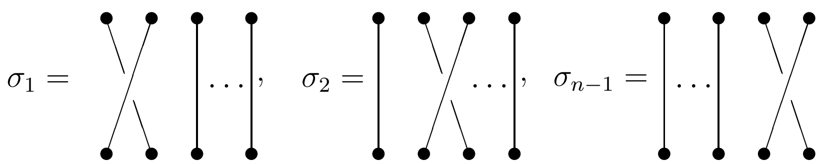
Let $\mathfrak{g}$ be the Lie algebra defined by a Cartan matrix $\mathbf{A}$ and let $\mathbf{B}$ be the corresponding braid group. Denote by $\mathbf{U}(\mathfrak{g})$ the quantized enveloping algebra of $\mathfrak{g}$ over the field $\mathbb{C}(v)$, and let V be the integrable $\mathbf{U}(\mathfrak{g})-$module. In [24] G. Lusztig defined a natural action of $\mathbf{B}$ on V which permutes the weight space of V according to the action of the Weyl group on the weights. This rather *general but different approach* allows us also to construct the irreducible representations of the braid group $\mathbf{B}$ (see [22]).

0. Definition of the Artin braid group B_n

$$B_n = \langle (\sigma_i)_{i=1}^{n-1}, \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad \sigma_i\sigma_j = \sigma_i\sigma_j, \quad \mid i-j \mid \geq 2\rangle.$$

$B_n = \pi_1(X)$ is the *fundamental group* π_1 of the *configuration space* $X = \{\mathbb{C}^n \setminus \Delta\}/S_n$ where $\Delta = \{(z_1, ..., z_n) \mid x_i = z_j \text{ for some } i \neq j\}$ and the group S_n act freely on $\mathbb{C}^n \setminus \Delta$ by permuting coordinates.

A **BRAID** on n strings *is a collection of curves in* $\mathbb{R}^3$ *joining* n *points in a horizontal plane to the* n *points directly below them on another horizontal plane*. Operation: *concatenation.*



Knot theory : Alexander, Jones, HOMFLYPT, Kauffman polynomials.
Respectively: *Temperley-Lieb, Hecke, BMW* algebras.
Geometry, physics etc.
Relation with the symmetric group $S_n : \sigma_i^2 = 1$

$$\sigma_i^2 = 1 \Rightarrow Sp\,(\rho(\sigma_i)) \subseteq \{-1, 1\}$$

$$Rep(S_n) \qquad Rep(B_n)?$$

$$\hat{S}_n = \{\textbf{Young diagrams}\}, \quad \textbf{Plancherel measure on } \hat{S}_n.$$

The *Young graph* explains how to *decompose the restriction* $\rho \mid_{S_{n-1}}$ of the representation $\rho \in \hat{S}_n$, etc.

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \Rightarrow Sp\,(\rho(\sigma_i)) = Sp\,(\rho(\sigma_{i+1})).$$

The *Hecke algebra* is defined by

$$H_n(q) = \langle \sigma_i\rangle_{i=1}^{n-1} \mid ...\sigma_i^2 = (q-1)\sigma_i + q\rangle, \quad p_2(\sigma_i) = 0,$$

hence $Sp\,(\rho(\sigma_i)) \subseteq \{-1, q\}$ and $H_n(q) \cong \mathbb{C}[S_n]$.

1. **Definition** $B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$.

2. **Homomorphism** $\rho : B_3 \mapsto \mathrm{SL}(2, \mathbb{Z})$,

$$\sigma_1 \mapsto \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \quad \sigma_2 \mapsto \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right), \quad \sigma_2 = (\sigma_1^{-1})^\sharp.$$

3. $B_3/Z(B_3) \simeq \mathrm{PSL}(2, \mathbb{Z}) \simeq \mathbb{Z}_2 * \mathbb{Z}_3$.

4. **P. Humphries result, Pascal's triangle**

$$\sigma_1 \mapsto \sigma_1(1, n), \ \sigma_2 \mapsto \sigma_2(1, n).$$

5. **Ferrand result** $\Phi_n, \ \Psi_n \in \mathrm{End}\, \mathbb{C}^n[X]$.

6. **Tubo-Wenzl example**

$$\sigma_1, \mapsto \sigma_1(1, n)\Lambda_n, \quad \sigma_2 \mapsto \Lambda_n^\sharp \sigma_2(1, n), \quad \Lambda_n \Lambda_n^\sharp = cI.$$

7. **Tubo - Wenzl classifications of** $B_3 -$ mod, dim$V \leq 5$.

8. **Generalizations**

$$\sigma_1 \mapsto \sigma_1^\Lambda(q, n) := \sigma_1(q, n) D_n(q)^\sharp \Lambda_n,$$

$$\sigma_2 \mapsto \sigma_2^\Lambda(q, n) := \Lambda_n^\sharp D_n(q) \sigma_2(q, n),$$

where $\sigma_2(q, n) = (\sigma_1^{-1}(q^{-1}, n))^\sharp, \ \Lambda_n = \mathrm{diag}(\lambda_r)_{r=0}^n, \ \Lambda_n \Lambda_n^\sharp = cI,$

$$D_n(q) = \mathrm{diag}(q_r)_{r=0}^n, \ q_r = q^{\frac{(r-1)r}{2}}, \ r, n \in \mathbb{N}.$$

9. **The connection between** $Rep(B_3)$ **and** $U_q(\mathfrak{sl}_2)$-mod.

10. **The Burau representation** $\rho_n : B_n \mapsto \mathrm{GL}_n(\mathbb{Z}[t, t^{-1}])$.

11. **Lowrence-Kramer representations**

12. **Generalization of 8 and 9 for** B_n.

13. **Formanek classifications of** $B_n -$ mod, for dim$V \leq n$.

1. $B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$.
2. $\rho : B_3 \mapsto \mathrm{SL}(2, \mathbb{Z})$,

$$\sigma_1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \sigma_2 \mapsto \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

3. $B_3/Z(B_3) \simeq \mathrm{PSL}(2, \mathbb{Z}) \simeq \mathbb{Z}_2 * \mathbb{Z}_3$.
Hint: **the Pascal triangle**, $\sigma_1 \mapsto \sigma_2$? $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$.

$$\sigma_1(1,2) := \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sigma_1^{-1}(1,2)^\sharp := \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{pmatrix}.$$

Notations the **central symmetry:**

$$A^\sharp := (A^t)^s, \quad A^\sharp = (a^\sharp_{ij}), \; a^\sharp_{ij} = a_{n-i,n-j},$$

$$\sigma_1 \mapsto \sigma_1(1,2), \quad \sigma_2 \mapsto \sigma_2(1,2) := \sigma_1^{-1}(1,2)^\sharp.$$

4. **P. Humphries**, [14] representations of B_3 in $\mathbb{C}^{n+1}$

$$\sigma_1 \mapsto \sigma_1(1,n), \quad \sigma_2 \mapsto \sigma_2(1,n) := \sigma_1^{-1}(1,n)^\sharp. \tag{1}$$

5. **Ferrand result**, [11]. $\Phi_n, \Psi_n \in \mathrm{End}\,\mathbb{C}^n[X] : \Phi_n\Psi_n\Phi_n = \Psi_n\Phi_n\Psi_n$.

$$(\Phi_n p)(X) := p(X+1), \quad (\Psi_n p)(X) := (1-X)^n p\left(X/(1-X)\right).$$

6. **Tubo-Wenzl example** [29]: representations $\sigma^\Lambda(1,n)$ of B_3 in $\mathbb{C}^{n+1}$

$$\sigma_1 \mapsto \sigma_1(1,n)\Lambda_n, \quad \sigma_2 \mapsto \Lambda_n^\sharp \sigma_2(1,n), \tag{2}$$

conditions on the complex diagonal matrix $\Lambda_n = \mathrm{diag}(\lambda_0, \lambda_1, ..., \lambda_n)$ are the following:

$$\Lambda_n \Lambda_n^\sharp = cI, \;\; c \in \mathbb{C}. \tag{3}$$

7. Tubo - Wenzl classifications of $B_3 - \mathrm{mod}$, $\dim V \leq 5$.

See [29]. Let V be a simple B_3 module of dimension $n = 2, 3$. Then there exist a basis for V for which σ_1 and σ_2 act as follows ($\lambda = (\lambda_k)_k$) for $n = 2$ and $n = 3$

$$\sigma_1^\lambda := \begin{pmatrix} \lambda_1 & \lambda_1 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \sigma_2^\lambda := \begin{pmatrix} \lambda_2 & 0 \\ -\lambda_2 & \lambda_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{pmatrix}, \tag{4}$$

$$\sigma_1 \mapsto \sigma_1^\lambda = \begin{pmatrix} \lambda_1 & \lambda_1\lambda_3\lambda_2^{-1}+\lambda_2 & \lambda_2 \\ 0 & \lambda_2 & \lambda_2 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad \sigma_2 \mapsto \sigma_2^\lambda := \begin{pmatrix} \lambda_3 & 0 & 0 \\ -\lambda_2 & \lambda_2 & 0 \\ \lambda_2 & -\lambda_1\lambda_3\lambda_2^{-1}-\lambda_2 & \lambda_1 \end{pmatrix}. \tag{5}$$

Let us set $D = \sqrt{\lambda_2\lambda_3/\lambda_1\lambda_4}$. All simple modules for $n = 4$ are the following:

$$\sigma_1 \mapsto \sigma_1^\lambda = \begin{pmatrix} \lambda_1 & (1+D^{-1}+D^{-2})\lambda_2 & (1+D^{-1}+D^{-2})\lambda_3 & \lambda_4 \\ 0 & \lambda_2 & (1+D^{-1})\lambda_3 & \lambda_4 \\ 0 & 0 & \lambda_3 & \lambda_4 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}, \tag{6}$$

$$\sigma_2 \mapsto \sigma_2^\lambda = \begin{pmatrix} \lambda_4 & 0 & 0 & 0 \\ -\lambda_3 & \lambda_3 & 0 & 0 \\ D\lambda_2 & -(D+1)\lambda_2 & \lambda_2 & 0 \\ -D^3\lambda_1 & (D^3+D^2+D)\lambda_1 & -(D^2+D+1)\lambda_1 & \lambda_1 \end{pmatrix}. \tag{7}$$

Let us set $\gamma = (\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5)^{1/5}$. All simple modules for $n = 5$ are the following:

$$\sigma_1 \mapsto \sigma_1^\lambda = \begin{pmatrix} \lambda_1 & (1+\frac{\gamma^2}{\lambda_2\lambda_4})(\lambda_2+\frac{\gamma^3}{\lambda_3\lambda_4}) & (\frac{\gamma^2}{\lambda_3}+\lambda_3+\gamma)(1+\frac{\lambda_1\lambda_5}{\gamma^2}) & (1+\frac{\lambda_2\lambda_4}{\gamma^2})(\lambda_3+\frac{\gamma^3}{\lambda_2\lambda_4}) & \frac{\gamma^3}{\lambda_1\lambda_5} \\ 0 & \lambda_2 & \frac{\gamma^2}{\lambda_3}+\lambda_3+\gamma & \frac{\gamma^3}{\lambda_1\lambda_5}+\lambda_3+\gamma & \frac{\gamma^3}{\lambda_1\lambda_5} \\ 0 & 0 & \lambda_3 & \frac{\gamma^3}{\lambda_1\lambda_5}+\lambda_3 & \frac{\gamma^3}{\lambda_1\lambda_5} \\ 0 & 0 & 0 & \lambda_4 & \lambda_4 \\ 0 & 0 & 0 & 0 & \lambda_5 \end{pmatrix}. \tag{8}$$

The formula for σ_2^λ was not given in [29].

8. Equivalence of Tuba-Wenzl's representations in the case $\dim \leq 5$ **and our representations**.

General formulas for $1 \leq n \leq 4$ gives us (we set $q_r = q^{\frac{(r-1)r}{2}}$):

$$\sigma_1 \mapsto \sigma_1^\Lambda := \sigma_1(q,n)\Lambda_n, \quad \sigma_2 \mapsto \sigma_2^\Lambda := \Lambda_n^\sharp \sigma_2(q,n),$$

$$\Lambda_n\Lambda_n^\sharp = \lambda_0\lambda_n\Lambda_n(q), \quad \Lambda_n(q) = q_n^{-1}D_n(q)D_n^\sharp(q),\ D_n(q) = \mathrm{diag}(q_r)_{r=0}^n,$$

$$\lambda_r\lambda_{n-r} = \lambda_0\lambda_n q^{-(n-r)r}, \quad 0 \leq r \leq n. \tag{9}$$

Let $n = 1$ we have

$$\sigma_1^\Lambda = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\Lambda_1, \quad \sigma_2^\Lambda = \Lambda_1^\sharp\left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right), \quad \Lambda_1 = \left(\begin{smallmatrix} \lambda_0 & 0 \\ 0 & \lambda_1 \end{smallmatrix}\right).$$

Let $n = 2$, conditions (9) gives us $\Lambda_2 = \mathrm{diag}(\lambda_r)_{r=0}^3$

$$\mathrm{diag}(\lambda_0\lambda_2, \lambda_1^2, \lambda_0\lambda_2) = \lambda_0\lambda_2\mathrm{diag}(1, q^{-1}, 1), \quad \text{so } q^{-1} = \lambda_1^2/\lambda_0\lambda_2.$$

$$\sigma_1^\Lambda(q,2) = \begin{pmatrix} 1 & 1+q & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\Lambda_2, \quad \sigma_2^\Lambda(q,2) = \Lambda_2^\sharp\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ q^{-1} & -(1+q^{-1}) & 1 \end{pmatrix}.$$

For $n = 3$ conditions (9) gives us $q^{-2} = \lambda_1\lambda_2/\lambda_0\lambda_3$ for $r = 1$.

$$\sigma_1(q,3) = \begin{pmatrix} 1 & 1+q+q^2 & 1+q+q^2 & 1 \\ 0 & 1 & 1+q & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda_0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix},$$

$$\sigma_2(q,3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ q^{-1} & -(1+q^{-1}) & 1 & 0 \\ -q^{-3} & q^{-1}(1+q^{-1}+q^{-2}) & -(1+q^{-1}+q^{-2}) & 1 \end{pmatrix}.$$

For $n = 4$ conditions (9) gives us $q^{-3} = \lambda_1\lambda_3/\lambda_0\lambda_4$ for $r = 1$ and $q^{-4} = \lambda_2^2/\lambda_0\lambda_4$ for $r = 2$.

$$\sigma_1(q) = \begin{pmatrix} 1 & (1+q)(1+q^2) & (1+q^2)(1+q+q^2) & (1+q)(1+q^2) & 1 \\ 0 & 1 & 1+q+q^2 & 1+q+q^2 & 1 \\ 0 & 0 & 1 & 1+q & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \ \Lambda = \begin{pmatrix} \lambda_0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & 0 & \lambda_4 \end{pmatrix},$$

$\sigma_2(q,4) = (\sigma_1^{-1}(q^{-1},4))^\sharp$.

$$\sigma_1 \mapsto \sigma_1(1,n)\Lambda_n, \quad \sigma_2 \mapsto \Lambda_n^\sharp \sigma_2(1,n), (2)$$
$$\Lambda_n = \mathrm{diag}(\lambda_r)_{r=0}^n, \quad \Lambda\Lambda^\sharp = cI, \ \ c \in \mathbb{C}, \quad (3)$$

8. Generalization of (2) for $q \neq 1$, with the condition (3)

$$\sigma_1 \mapsto \sigma_1^\Lambda(q,n) := \sigma_1(q,n)D_n^\sharp(q)\Lambda_n, \ \sigma_2 \mapsto \sigma_2^\Lambda(q,n) := \Lambda_n^\sharp D_n(q)\sigma_2(q,n), \quad (10)$$

$$\sigma_2(q,n) := \sigma_1^{-1}(q^{-1},n)^\sharp, \ D_n(q) = \mathrm{diag}(q_r)_{r=0}^n, \ q_r = q^{\frac{(r-1)r}{2}}, \qquad (11)$$

where $q-$ *binomial coefficients* or *Gaussian polynomials* are defined as follows

$$\left(\begin{smallmatrix} n \\ k \end{smallmatrix}\right)_q := \frac{(n)!_q}{(k)!_q(n-k)!_q}, \quad \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q := \frac{[n]!_q}{[k]!_q[n-k]!_q} \qquad (12)$$

corresponding to two forms of $q-$*natural numbers*, defined by

$$(n)_q := \frac{q^n - 1}{q - 1}, \quad [n]_q := \frac{q^n - q^{-1}}{q - q^{-1}}. \qquad (13)$$

Theorem 1 [1] *The formulas (10) $\sigma_1 \mapsto \sigma_1^\Lambda(q,n)$, $\sigma_2 \mapsto \sigma_2^\Lambda(q,n)$ give the representation of B_3.*

Theorem 2 [1] *The representation $\sigma^\Lambda(q,n)$ defined by (10) generalize the Tubo-Wenzl representations for arbitrary $n \in \mathbb{N}$.*

Definition. *We say that the representation is* **subspace irreducible** *or* **ireducible** *(resp.* **operator irreducible***) when there no nontrivial invariant close* **subspaces** *for all operators of the representation (resp. there no nontrivial bounded* **operators** *commuting with all operators of the representation).*

Let us define for n, r, q, λ such that $n \in \mathbb{N}, \ 0 \leq r \leq n, \ \lambda \in \mathbb{C}^{n+1}, \ q \in \mathbb{C}$ the following operators

$$F_{r,n}(q,\lambda) = \exp_{(q)}\left(\sum_{k=0}^{n-1}(k+1)_q E_{kk+1}\right) - q_{n-r}\lambda_r(D_n(q)\Lambda_n^\sharp)^{-1}, \qquad (14)$$

where $\exp_{(q)} X = \sum_{m=0}^\infty X^m/(m)!_q$. For the matrix $C \in \mathrm{Mat}(n+1, \mathbb{C})$ we denote by

$$M^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_r}(C), \ \text{(resp. } A^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_r}(C)), \ 0 \leq i_1 < \dots < i_r \leq n, \ 0 \leq j_1 < \dots < j_r \leq n$$

its minors (resp. the cofactors) with $i_1, i_2, \dots, i_r$ rows and $j_1, j_2, \dots, j_r$ columns.

Theorem 3 [1] *The representation of the group B_3 defined by (10) have the following properties:*
1) for $q = 1,\ \Lambda_n = 1$, it is subspace irreducible *in arbitrary dimension $n \in \mathbb{N}$;*
2) for $q \neq 1,\ \Lambda_n = \mathrm{diag}(\lambda_k)_{k=0}^n \neq 1$ it is operator irreducible *if and only if for any $0 \leq r \leq \left[\frac{n}{2}\right]$ there exists $0 \leq i_0 < i_i < ... < i_r \leq n$ such that*

$$M^{i_0 i_i \dots i_{n-r-1}}_{r+1 r+2 \dots n}(F^s_{r,n}(q,\lambda)) \neq 0; \tag{15}$$

3) for $q \neq 1,\ \Lambda_n = 1$ it is subspace irreducible if and only if $(n)_q \neq 0$.
The representation has $[\frac{n+1}{2}] + 1$ free parameters.

9. The connection between $Rep(B_3)$ and $U_q(\mathfrak{sl}_2)$-mod.

The algebra $U(\mathfrak{sl}_2)$ is the associative algebra generated by three generators X, Y, H with the relations (7).

$$[H, X] = 2X,\ [H, Y] = -2Y,\ [X, Y] = H, \tag{16}$$

$$X = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \quad Y = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), \quad H = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \quad \text{in} \quad \mathfrak{sl}_2.$$

$U_q(\mathfrak{sl}_2)$ is the algebra generated by four variables E, F, K, K^{-1} with the relations

$$KK^{-1} = K^{-1}K = 1, \tag{17}$$

$$KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2}F, \tag{18}$$

$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}} = \frac{q^H - q^{-H}}{q - q^{-1}}. \tag{19}$$

Comultiplication Δ, counit ε and antipod S are as follows:

$$\Delta(E) = E \otimes K + 1 \otimes E, \quad \Delta(F) = F \otimes 1 + K^{-1} \otimes F, \quad \Delta(K) = K \otimes K,$$

$$S(K) = K^{-1},\ S(E) = -EK^{-1},\ S(F) = -KF,$$

$$\varepsilon(K) = 1,\ \varepsilon(E) = \varepsilon(F) = 0.$$

All finite-dimensional $U-$module V being the highest weight module of highest weight λ are of the following form (see Kassel, [17, TheoremV.4.4.])

$$\rho(n)(X) = \begin{pmatrix} 0 & n & 0 & \dots & 0 \\ 0 & 0 & n-1 & \dots & 0 \\ & & & \dots & \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad \rho(n)(Y) = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 2 & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & \dots & n & 0 \end{pmatrix},$$

$$\rho(n)(H)=\begin{pmatrix} n & 0 & \dots & 0 & 0\\ 0 & n-2 & \dots & 0 & 0\\ & & \dots & & \\ & & \dots & -n+2 & 0\\ 0 & 0 & \dots & 0 & -n \end{pmatrix}.$$

where $\lambda=\dim(V)-1\in\mathbb{N}$.

All finite-dimensional U_q-module V being the highest weight module of highest weight λ are of the following form (see Kassel, [17, Theorem VI.3.5.])

$$\rho_{\varepsilon,n}(E)=\varepsilon\begin{pmatrix} 0 & [n] & 0 & \dots & 0\\ 0 & 0 & [n-1] & \dots & 0\\ & & & \dots & \\ 0 & 0 & 0 & \dots & 1\\ 0 & 0 & 0 & \dots & 0 \end{pmatrix},\quad \rho_{\varepsilon,n}(F)=\begin{pmatrix} 0 & 0 & \dots & 0 & 0\\ 1 & 0 & \dots & 0 & 0\\ 0 & [2] & \dots & 0 & 0\\ & & \dots & & \\ 0 & 0 & \dots & [n] & 0 \end{pmatrix},$$

$$\rho_{\varepsilon,n}(K)=\varepsilon\begin{pmatrix} q^n & 0 & \dots & 0 & 0\\ 0 & q^{n-2} & \dots & 0 & 0\\ & & \dots & & \\ & & \dots & q^{-n+2} & 0\\ 0 & 0 & \dots & 0 & q^{-n} \end{pmatrix},$$

where $\varepsilon=\pm1,\ \lambda=\varepsilon q^n$ and $n\in\mathbb{N}$.

The main observation is the following:

$$\begin{pmatrix} 1 & 2 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1 \end{pmatrix}=\exp\begin{pmatrix} 0 & 2 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0 \end{pmatrix},\quad \begin{pmatrix} 1 & (2)_q & 1\\ 0 & 1 & (1)_q\\ 0 & 0 & 1 \end{pmatrix}=\exp_{(q)}\begin{pmatrix} 0 & (2)_q & 0\\ 0 & 0 & (1)_q\\ 0 & 0 & 0 \end{pmatrix},$$

where

$$\begin{pmatrix} 0 & (2)_{q^2} & 0\\ 0 & 0 & (1)_{q^2}\\ 0 & 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & [2]_q & 0\\ 0 & 0 & [1]_q\\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} q^2 & 0 & 0\\ 0 & q & 0\\ 0 & 0 & 1 \end{pmatrix},\ \exp_{(q)}X:=\sum_{m=0}^{\infty}\frac{1}{(m)!_q}X^m.$$

Theorem 4 *For $q=1$ holds*

$$\sigma_1(1,n)=\exp\left(\rho(n)(X)\right),\quad \sigma_2(1,n)=\exp\left(\rho(n)(-Y)\right). \tag{20}$$

Theorem 5 *For $q\neq1$ we have*

$$\sigma_1(q^2,n)D_n^\sharp(q^2)=\exp_{(q^2)}\left(q^{n/2}\rho_{1,n}(EK^{1/2})\right)D_n^\sharp(q^2), \tag{21}$$

$$D_n(q^2)\sigma_2(q^2,n)=\exp_{(q^2)}\left(-q^{n/2}\rho_{1,n}(FK^{-1/2})\right)D_n(q^2). \tag{22}$$

Proof. The two forms of $q-$natural numbers are connected as follows (see Kassel, [17])

$$[n]=q^{-(n-1)}(n)_{q^2},\quad [n]!=q^{-(n-1)n/2}(n)!_{q^2} \tag{23}$$

$$\begin{pmatrix} 0 & (n) & 0 & ... & 0 \\ 0 & 0 & (n-1) & ... & 0 \\ & & & ... & \\ 0 & 0 & 0 & ... & (1) \\ 0 & 0 & 0 & ... & 0 \end{pmatrix} = \begin{pmatrix} 0 & [n] & 0 & ... & 0 \\ 0 & 0 & [n-1] & ... & 0 \\ & & & ... & \\ 0 & 0 & 0 & ... & [1] \\ 0 & 0 & 0 & ... & 0 \end{pmatrix} \mathrm{diag}(q^n, q^{n-1}, ..., 1)$$

$= q^{n/2}\rho_{1,n}(EK^{1/2})$, and

$$\begin{pmatrix} 0 & 0 & ... & 0 & 0 \\ (1) & 0 & ... & 0 & 0 \\ 0 & (2) & ... & 0 & 0 \\ & & ... & & \\ 0 & 0 & ... & (n) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & ... & 0 & 0 \\ [1] & 0 & ... & 0 & 0 \\ 0 & [2] & ... & 0 & 0 \\ & & ... & & \\ 0 & 0 & ... & [n] & 0 \end{pmatrix} \mathrm{diag}(1, q, ..., q^{n-1}, q^n)$$

$= q^{n/2}\rho_{1,n}(FK^{-1/2})$, since

$$\mathrm{diag}(1, q, ..., q^{n-1}, q^n) = q^{n/2}\rho_{1,n}(K^{-1/2})$$

and

$$\mathrm{diag}(q^n, q^{n-1}, ..., 1) = q^{n/2}\rho_{1,n}(K^{1/2}).$$

Al last we conclude that

$$\begin{pmatrix} 0 & (n) & 0 & ... & 0 \\ 0 & 0 & (n-1) & ... & 0 \\ & & & ... & \\ 0 & 0 & 0 & ... & (1) \\ 0 & 0 & 0 & ... & 0 \end{pmatrix} = q^{n/2}\rho_{1,n}(EK^{1/2}),$$

$$\begin{pmatrix} 0 & 0 & ... & 0 & 0 \\ (1) & 0 & ... & 0 & 0 \\ 0 & (2) & ... & 0 & 0 \\ & & ... & & \\ 0 & 0 & ... & (n) & 0 \end{pmatrix} = q^{n/2}\rho_{1,n}(FK^{-1/2}).$$

Further we observe that

$$X \otimes I + I \otimes X \mid_{S^2(\mathbb{C}^2)} = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \otimes I + I \otimes \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \mid_{S^2(\mathbb{C}^2)} = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Delta\rho(1)(X) \mid_{S^2(\mathbb{C}^2)} = \rho(2)(X),$$

$$(I + X) \otimes (I + X) = \exp(\Delta(X)), \quad \sigma_1(1,1) \otimes \sigma_1(1,1) \mid_{S^2(\mathbb{C}^2)} = \sigma(1,2).$$

Lemma 6 *We have for $q \neq 1$*

$$\rho_{1,n} = \Delta^{n-1}\rho_{1,1} \mid_{S^{n,q}(\mathbb{C}^2)}, \tag{24}$$

where $S^{n,q}(\mathbb{C}^2)$ is $q-$symmetric tensor power of $\mathbb{C}^2$.

Proof. For $n = 1$ we have the following operators

$$\rho_{1,1}(E) = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \quad \rho_{1,1}(F) = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), \quad \rho_{1,1}(K) = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} = q^H.$$

For $n = 2$ we get

$$\rho_{1,2}(E) = \begin{pmatrix} 0 & [2] & 0 \\ 0 & 0 & [1] \\ 0 & 0 & 0 \end{pmatrix}, \; \rho_{1,2}(F) = \begin{pmatrix} 0 & 0 & 0 \\ [1] & 0 & 0 \\ 0 & [2] & 0 \end{pmatrix}, \; \rho_{1,2}(K) = \begin{pmatrix} q^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q^{-2} \end{pmatrix}$$

We have $\Delta(\rho_{1,1}(E)) =$

$$\rho_{1,1}(E) \otimes \rho_{1,1}(K) + 1 \otimes \rho_{1,1}(E) = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \otimes \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} + \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$$

$$= \begin{pmatrix} 0 & 0 & q & 0 \\ 0 & 0 & 0 & q^{-1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & q & 0 \\ 0 & 0 & 0 & q^{-1} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Further $\Delta(\rho_{1,1}(F)) =$

$$\rho_{1,1}(F) \otimes 1 + \rho_{1,1}(K^{-1}) \otimes \rho_{1,1}(F) = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) + \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix} \otimes \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ q^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & q & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ q^{-1} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & q & 0 \end{pmatrix}$$

and

$$\Delta(\rho_{1,1}(K)) = \rho_{1,1}(K) \otimes \rho_{1,1}(K) = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} \otimes \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} = \begin{pmatrix} q^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q^{-2} \end{pmatrix}.$$

In the $q-$*symmetric basis* of the submodule $S^{2,q}(\mathbb{C}^2)$ of the module $\mathbb{C}^2 \otimes \mathbb{C}^2$

$$e^{s,q}_{00} = e_0 \otimes e_0, \quad e^{s,q}_{01} = q^{-1} e_0 \otimes e_1 + e_1 \otimes e_0, \quad e^{s,q}_{11} = e_1 \otimes e_1$$

the operator $\Delta(\rho_{1,1}(E))$ has the following form:

$$\Delta(\rho_{1,1}(E)) \mid_{S^{2,q}(\mathbb{C}^2)} = \begin{pmatrix} 0 & [2] & 0 \\ 0 & 0 & [1] \\ 0 & 0 & 0 \end{pmatrix}.$$

The basis in the space $\mathbb{C}^2 \otimes \mathbb{C}^2$ is generated by vectors e_{kn}, $0 \leq k, n \leq 1$ where $e_{kn} = e_k \otimes e_n$. Operator $\Delta(\rho_{1,1}(E))$ acts as follows $e_{00} \mapsto 0$, $e_{01} \mapsto e_{00}$, $e_{10} \mapsto q e_{00}$, $e_{11} \mapsto \; q^{-1} e_{01} + e_{10}$, hence $e^{s,q}_{00} \mapsto 0$,

$$e^{s,q}_{01} = q^{-1} e_{01} + e_{10} \mapsto (q + q^{-1}) e_{00} = [2] e^{s,q}_{00}, \; e^{s,q}_{11} \mapsto \; q^{-1} e_{01} + e_{10} = e^{s,q}_{01}.$$

Similarly we get

$$\Delta(\rho_{1,1}(F)) \mid_{S^{2,q}(\mathbb{C}^2)} = \begin{pmatrix} 0 & 0 & 0 \\ [1] & 0 & 0 \\ 0 & [2] & 0 \end{pmatrix}, \quad \Delta(\rho_{1,1}(K)) \mid_{S^{2,q}(\mathbb{C}^2)} = \begin{pmatrix} q^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q^{-2} \end{pmatrix}.$$

hence (24) holds for $n = 2$. For $n > 2$ the proof is similar.

10. The Burau representation $\rho : B_n \mapsto \mathrm{GL}_n(\mathbb{Z}[t, t^{-1}])$ is defined for a non-zero complex number t by

$$\sigma_i \mapsto \beta_i = I_{i-1} \oplus \left(\begin{smallmatrix} 1-t & t \\ 1 & 0 \end{smallmatrix}\right) \oplus I_{n-i-1}$$

where $1 - t$ is the (i, i) entry. Representation ρ splits into 1-dimesional and $n-1-$dimensional irreducible representations, known as *reduced Burau representation* $\overline{\rho} : B_n \mapsto \mathrm{GL}_{n-1}(\mathbb{Z}[t, t^{-1}])$

$$\sigma_1 \mapsto b_1 = \left(\begin{smallmatrix} -t & 0 \\ -1 & 1 \end{smallmatrix}\right) \oplus I_{n-3}, \quad \sigma_{n-1} \mapsto b_{n-1} = I_{n-3} \oplus \left(\begin{smallmatrix} 1 & -t \\ 0 & -t \end{smallmatrix}\right),$$

$$\sigma_i \mapsto b_i = I_{i-2} \oplus \begin{pmatrix} 1 & -t & 0 \\ 0 & -t & 0 \\ 0 & -1 & 1 \end{pmatrix} \oplus I_{n-i-2}, \ 2 \le i \le n-2.$$

Problem. Whether the reduced Burau representation $\overline{\rho} : B_n \mapsto \mathrm{GL}_{n-1}(\mathbb{Z}[t, t^{-1}])$ is *faithful* ?

YES for $n = 3$ (Birman [8]). NO for $n \ge 9$ Moody [25] Long and Paton [23], Bigelow [6] improved further for $n \ge 5$.

Open problem: Whether the reduced Burau representation of $B_4 \mapsto \mathrm{GL}_3(\mathbb{Z}[t, t^{-1}])$

$$b_1 = \begin{pmatrix} -t & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ b_2 = \begin{pmatrix} 1 & -t & 0 \\ 0 & -t & 0 \\ 0 & -1 & 1 \end{pmatrix}, \ b_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -t \\ 0 & 0 & -t \end{pmatrix}$$

is **faithful**

11. Lowrence-Kramer representations, [20]

$$\lambda : B_n \mapsto \mathrm{GL}_m(\mathbb{Z}[t^{\pm 1}, q^{\pm 1}]), \quad m = n(n-1)/2.$$

The basis in the space $\mathbb{C}^{n(n-1)/2}$ is x_{ik}, $1 \le i < k \le n$.

Faithfulness for all n, Bigelow [7], Kramer [21] $\Rightarrow$ B_n **is a linear group for all** n.

$$
\begin{array}{rll}
\sigma_k x_{k,k+1} = & tq^2 x_{k,k+1} & \\
\sigma_k x_{ik} = & (1-q)x_{ik} + qx_{i,k+1} & \text{for } i < k \\
\sigma_k x_{i,k+1} = & x_{ik} + tq^{k-i+1}(q-1)x_{k,k+1} & \text{for } i < k \\
\sigma_k x_{kj} = & tq(q-1)x_{k,k+1} + qx_{k+1,j} & \text{for } k+1 < j \\
\sigma_k x_{k+1,j} = & x_{kj} + (1-q)x_{k+1,j} & \text{for } k+1 < j \\
\sigma_k x_{ij} = & x_{ij} & \text{for } i < j < k \text{ or } k+1 < i < j \\
\sigma_k x_{ij} = & x_{ij} + tq^{k-i}(q-1)^2 x_{k,k+1} & \text{for } i < k < k+1 < j
\end{array}
$$

12. Generalization of 8 and 9 for B_n**.** For $n = 4$ and $t = -1$ we have $\overline{\rho}_4 : B_4 \mapsto \mathrm{SL}(3, \mathbb{Z})$

$$
b_1 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; b_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \; b_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.
$$

$$
b_1 = \exp(-F_1),\, b_2 = \exp(E_1 - F_2), b_3 = \exp(E_2).
$$

We can show that the symmetric powers $b_i \otimes b_i \mid_S$ are the following

$$
b_1 \otimes b_1 \mid_S = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \; b_2 \otimes b_2 \mid_S = \begin{pmatrix} 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{pmatrix},
$$

$$
b_3 \otimes b_3 \mid_S = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.
$$

We have for $n = 5$ and $t = -1$ $b^{(5)} : B_5 \mapsto \mathrm{SL}(4, \mathbb{Z})$

$$
b_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; b_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; b_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \; b_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

Let $\overline{\rho} : B_n \mapsto \mathrm{SL}_{n-1}(\mathbb{Z})$ be the *reduced Burrau representation* for $t = -1$.

The *quantum group* $U_q(\mathfrak{sl}_{\mathfrak{n-1}})$ is the algebra generated by $4(n-1)$ variables $E_i,\ F_i,\ K_i,\ K_i^{-1}$ with relations as (17)–(19). Let

$$
\rho_m : U_q(\mathfrak{sl}_{\mathfrak{n-1}}) \mapsto \mathrm{End}(\mathbb{C}^{\mathfrak{m}})
$$

be the highest weight $U_q(\mathfrak{sl}_{n-1})-$module. *Then*

$$\sigma_1 \mapsto \exp(-\rho_m(F_1)),\ \sigma_k \mapsto \exp(\rho_m(E_{k-1}-F_k)),\ \ \sigma_n \mapsto \exp(\rho_m(E_{n-1})).$$

gives the representation of B_n *for* $q=1$ (see (20)).

For $q \neq 1$ we can obtain formulas similar to (21)–(22).

13. Formanek classifications of $B_n - \mathrm{mod}$**, for** $\mathrm{dim}V \leq n$.

In [12] E.Formanek et al. gave the *complete classification* of all *simple representations of* B_n for *dimension* $\leq n$.

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